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CONVEXITY ADJUSTMENT FOR CONSTANT MATURITY SWAPS AND LIBOR-IN-ARREARS BASIS SWAPS[1 2]

INTRODUCTION

The Constant Maturity Swap or Treasury (CMS or CMT) market is large and active. The difficulty of evaluating the implicit convexity cost, however, makes the markets more opaque than would otherwise be the case. This note lays out a practical method for calculating the value of the convexity adjustment for the linear CMS/CMT and LIBOR-in-arrears payments.

Both CMT/CMS and LIBOR-in-arrears swaps share the characteristic that the payment on one side of the swap is linear with respect to its index while the offsetting hedge is convex. The linearity of the payment (relative to the convex hedges) imposes a cost that leads to the "convexity adjustment" made to the linear payment. The basis of the approach used here is to

1. Find for each reset date the equivalent martingale measure (forward measure using a zero bond as numeraire) which makes the PV of a traded swap equal to its market value. In practical terms, this reduces to finding the "adjusted mean" of the rate distribution as of the reset date. For log-normally distributed rates, there are approximations which make this fast.
2. The same forward measure is then used to calculate the PV of the CMS/CMT swap or the LIBOR-in-arrears payment (which is not actively traded). This PV incorporates the convexity cost of the linear payment relative to its (traded) convex hedge.
3. This process is repeated for each payment of the CMS/CMT or LIBOR-in-arrears swap, and thus for the whole instrument.

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[1] An earlier version appeared in Derivatives Quarterly winter 1995.
[2] I would like to thank Andy Morton, Stuart Turnbull, and Alan Brazil for comments. Naturally, errors and ommisions are my own.

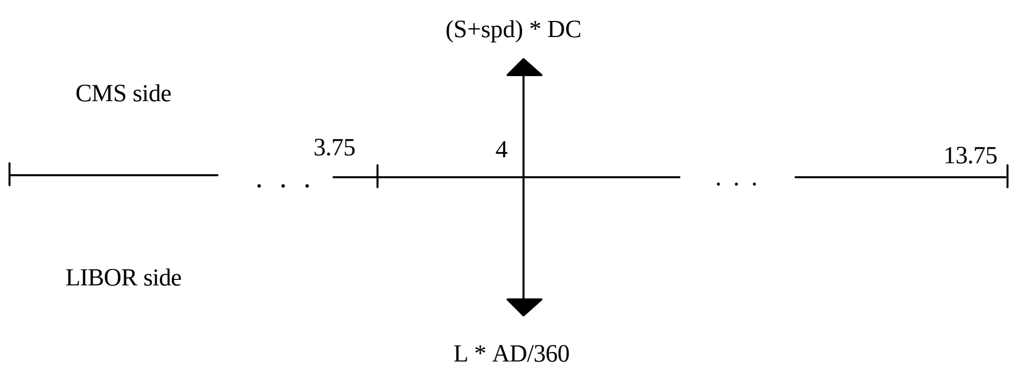
The result is the convexity-adjusted PV of the instrument. The convexity adjustment can be measured on a reset-by-reset basis as a spread equal to the difference between the adjusted mean (from the equivalent martingale measure) and the forward rate.

DESCRIPTION OF CMS/CMT AND LIBOR-IN-ARREARS

A constant maturity swap is a variation on a standard basis swap. One side is LIBOR as usual, but the other side is determined using a rate such as the 5 year swap rate or the 5 year Treasury rate. Constant maturity swaps can use a variety of indexes. The Federal Reserve's constant maturity Treasury (CMT) index is the most common, with constant maturity swap (CMS) rates being the next most common.

As an example, take a 10 year CMS swap receiving the 10 year CMS rate and paying standard LIBOR. Resets and payments are made quarterly. Figure 1 shows diagramatically the payments made at year four. The pay side is standard LIBOR: The LIBOR rate is set at three years nine months and paid in arrears at year four. The receive side is the 10 year swap rate set at three years nine months (the rate applies from year 3.75 through 13.75) and paid in arrears at year four with a 30/360 day-count fraction (DC) (plus or minus a spread). The key factor is that the CMS side pays quarterly, but using as the index a 10 year swap rate.

Figure 1 - Payments on CMS Swap at Year 4



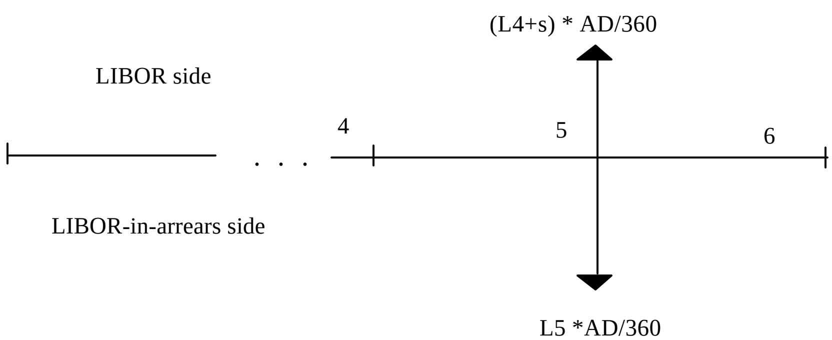
A CMS/CMT swap trades at a spread to floating LIBOR. The spread is a result of:

1. **Curve**: For an upward sloping yield curve the CMS/CMT rate will be higher than LIBOR, and one would receive CMS/CMT less a spread.
2. **Day Count Basis**: The CMS/CMT side often pays quarterly but uses a semi-annually quoted rate; this introduces an implicit spread.
3. **Convexity**: The linearity of the CMS/CMT payment combined with the convexity of hedge instruments leads to a benefit to receiving CMS/CMT which must be reflected in the spread.

The first two effects are straight-forward, but the convexity adjustment is more difficult to evaluate.

A LIBOR-in-arrears swap is also a variation on a standard basis swap. Here, LIBOR is the index on both sides, but the LIBOR-in-arrears rate is both set and paid in arrears rather than set up-front and paid in arrears (as for standard LIBOR). Side 1 payments are standard LIBOR (plus a spread) set up-front and paid in arrears. To make things concrete, say one agrees to a 5 year swap (on annual LIBOR) paying LIBOR-in-arrears and receiving LIBOR. Focus for now on just the last payment, at year 5. The payments would be as shown in figure 2.

Figure 2 - Payments on LIBOR-in-arrears Swap at Year 5



At year 5 one receives $(\underline{L}_4+s)\cdot AD/360$, where italics denote a random variable. This is year 4 LIBOR set up-front but paid in arrears. At year 5 one pays LIBOR-in-arrears (set and paid in arrears): $\underline{L}_5 \cdot AD/360$, where $\underline{L}_5$ is the one year LIBOR rate set at year 5.

USING CMS/CMT SWAPS

In terms of size, the CMT market is larger than CMS. The primary reason is the liquidity and depth of the Treasury market: Treasuries provide a benchmark against which many trades are measured and off which many instruments are priced. Three examples of using CMT swaps should suffice to show some of their importance.

The first example concerns floating rate bonds indexed to CMT. An investor who buys a CMT floater can use a swap to synthetically replicate a LIBOR floater. Many mortgage-backed CMO (collateralized mortgage obligation) floating rate tranches use CMT as the floating rate index. Many investors, however, fund at LIBOR (or a spread to LIBOR) and so may wish to receive LIBOR instead of CMT. By entering into a swap where the investor pays CMT and receives LIBOR, the investor can buy a CMT floater but receive LIBOR. Recognizing the effect of convexity in the CMT index is important both in valuing the relative price of the original CMT floater and in evaluating the CMT/LIBOR swap. Convexity effects raise the value of receiving the CMT index relative to receiving LIBOR. This means that a floating bond on which one receives CMT is actually more valuable than indicated by simply pricing off the forward curve (with no convexity adjustment). On the swap side, the swap to pay CMT and receive LIBOR will have the convexity effect priced in; the CMT payment will generally be lower than that implied by the forward curve (again with no convexity adjustment).

The second example, also from the mortgage backed market, concerns CMT inverse floaters. The coupon on a CMT inverse might be 12% less the CMT index, with a floor of 0%. An investor wishing to avoid the coupon risk (that the coupon falls as the CMT rate rises) can use a combination of a CMT swap and standard interest rate swaps convert the inverse floater to a LIBOR floater. (A CMT cap could also be sold, which would monetize the value of the coupon floor.) Once again, the convexity effect enters into the valuation of the original inverse floater: the true value of paying CMT is lower than the value implied by the forward curve (without adjusting for convexity). In other words, the true value of the inverse floater is less than that implied by pricing the CMT index off the forward curve (ignoring the convexity adjustment).

The third example is using CMS/CMT swaps for taking positions along the yield curve. Receiving the CMS or CMT index on a swap will benefit from the yield curve remaining steeper than implied by the forward curve; i.e. it will allow the investor to take a view on the shape of the yield curve. If the CMS or CMT rate remains above that implied by the forward curve, and investor receiving the index will benefit. The spread on the swap will price in the steepness of the forward curve, but will also price the value of convexity. Receiving the CMS or CMT index has value relative to buying forward swaps or bonds. Properly pricing the convexity effect is important in evaluating the merits of using a CMS/CMT swap to implement a curve view.

HOW CONVEXITY ENTERS

Consider again figure 1, a single reset on the CMS leg of a swap receiving the 10 year CMS rate. If the CMS rate changes by 1bp the profit or loss is 1¢.[3] To hedge this, one would receive fixed on $14.39 notional of a 10 year swap: when the 10 year swap rate changes by 1bp the profit or loss would also be 1¢.

The convexity adjustment arises because the CMS/CMT payment is linear in the index while the hedge, a standard swap, is convex. Although the hedge matches when the change is only 1bp, convexity enters when the change is larger than 1bp. If the CMS index rises by 100bp the profit on the CMS leg is $1. The hedging swap, however, would lose $0.96. If the CMS index falls by 100bp the loss on the CMS leg is $1 while the profit on the swap would be $1.05. Figure 3 shows the hedge mismatch graphically.

Using a standard swap is not a perfect hedge for a CMS swap. The best one can do is to receive fixed on a forward 10 year swap in a risk-weighted amount. The value as of the reset date (forward value or FV) for the 10 year swap can be written as:

$$FV(S_{10} \, ; \underline{S}_{10})$$

where

S_{10} = fixed coupon of forward swap (fixed at the time the hedge is put in place, but initially set at the forward par swap rate)

$\underline{S}_{10}$ = par swap yield at forward date (random as of today) .

The hedge ratio (the notional amount of the forward swap) is determined by setting the first derivative of the forward swap (evaluated at the forward swap rate, shown on the left-hand-side) equal to the derivative of the linear CMS payment (shown on the right-hand-side):

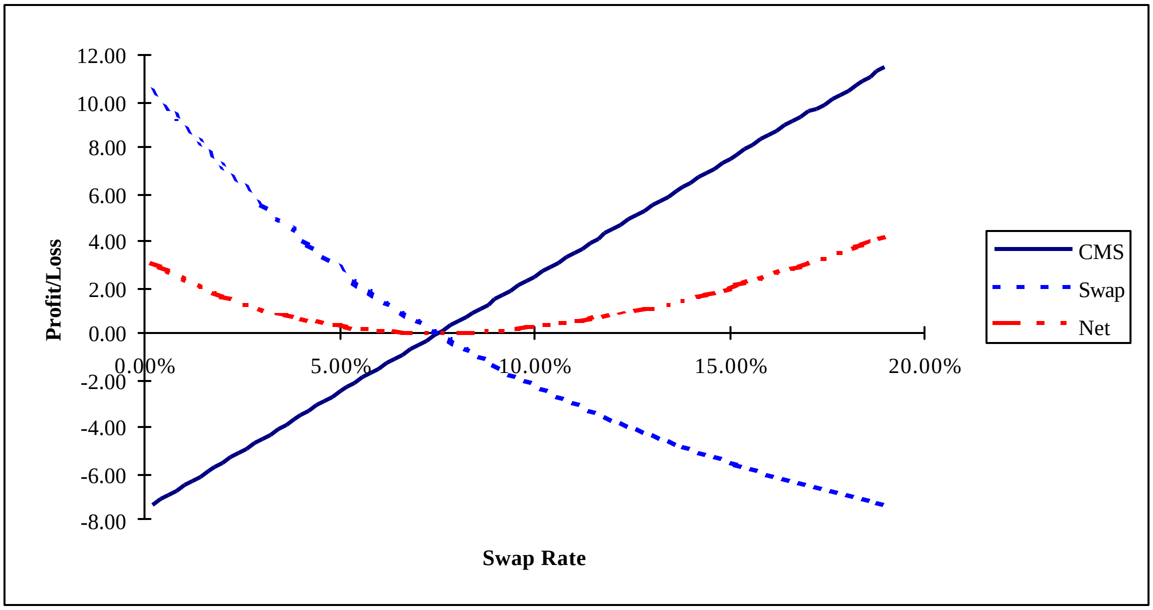
$$\partial FV(S_{10} \, ; S_{10}) / \partial S_{10} \quad = \quad \partial \, S_{10} / \partial \, S_{10}$$

[3] This assumes a flat 7.5%sab curve and a notional of $100 on the CMS swap.

Note, however, that the hedge is convex (has a non-zero second derivative) while the linear CMS payment is not convex (it is linear and has a zero second derivative).

Figure 3 shows the net profit and loss of a portfolio of receiving the CMS rate and hedging by receiving fixed on a standard swap. The profit from this strategy is always non-negative; it is impossible to lose money by receiving the CMS rate and hedging with a standard swap. Given that there are few (if any) free lunches in the financial markets, this is unlikely. The result is that one receives the CMS rate **less** a spread (over-and-above the spread resulting from the shape of the curve and day basis). This spread is the convexity spread.

Figure 3 - Profit & Loss on Portfolio: Receiving CMS, Receive Fixed on Swap



SIZE OF CONVEXITY COST - CMS EXAMPLE

As an example of the size of the convexity adjustment, consider a 10 year swap against 10 year CMS. For the last quarterly reset, in 9.75 years with payment in 10 years, the convexity spread would be about 57bp or 6.8bp up-front cost. For the whole swap the spread resulting from convexity (over and above any spread resulting from slope of the curve and day basis) would be about 25bp.

The cost imposed by the convexity of hedges relative to the linearity of the payment can be large. The cost is larger

1. The further out is the payment - e.g. the convexity cost of a reset in 10 years is greater than the cost of a reset in one year
2. The larger the duration of the underlying hedge instrument - e.g. the cost is larger for payments against 10 year CMS than one year CMS
3. The higher the volatility of the forward yield of the underlying hedge instrument.

Figure 4 shows the effect of time to payment, duration of the underlying hedge instrument, and volatility of the forward yield.

Figure 4 - Effect of Time to Payment, Duration, an Volatility.
Single Quarterly Payment, 7.5% Forward Curve

	Spread(bp)	Up-front cost(bp)
BASE CASE	57	6.8
Reset in 9.75yrs, payment in 10yrs		
Index = 10yr CMS rate		
Volatility = 15%		
Reset in 1 yr, payment in 1.25yrs	5	1.2
All else as in base case		
Index = 1yr CMS rate	6	0.8
All else as in base case		
Volatility = 10%	24	2.9
All else as in base case		

INTRODUCTION TO EQUIVALENT MARTINGALE APPROACH

It is clear that the hedge to offset the linear CMS payment is costly (relative to the linear payment). In effect, one ends up paying more than the implied forward rate on the CMS leg because of volatility in the swap rate. This convexity cost should enter into the pricing of the linear CMS payment. The problem is to quantify the convexity cost.

There are two solutions which immediately come to mind. The first is to take a term-structure model that produces an arbitrage-free distribution of swap rates as of each payment date (a model such as Heath, Jarrow, and Morton (1992) or Black, Derman, and Toy (1990)) and discount the linear CMS payments back using the model. This will, by necessity, value the implicit cost of the convexity (or lack of it) in the CMS payment. The only draw-back of this method is the numerical complexity of calibrating the model and valuing the payment using a tree, lattice, or monte-carlo method.

The second approach, and that discussed here, is to find the equivalent martingale measure that makes the expected value of the forward swap equal to the traded price of the forward swap, and then take the expectation of the CMS payment over this measure. Although this sounds complicated, it is actually rather straight-forward.[4]

EQUIVALENT MARTINGALE APPROACH - PAYMENT UP FRONT

Assume to start with that the reset and payment are both up front, so that the rate reset at 3.75 years is also paid at 3.75 years.[5] Then the linear CMS payment will be hedged with a 3.75 year forward 10 year swap. The first step is to value the hedging swap under the equivalent martingale measure. Using a pure discount (zero) bond maturing at year 3.75 as numeraire, one can show there exists a measure such that the time-t value of the forward bond is a martingale. That is,

$$FV(S_{10}\ ;\ \underline{curve}_t) \div \underline{B}_{3.75}(t)$$

is a martingale. By the fact that $FV \div B$ is a martingale,

(1) $$FV(S_{10}\ ;\ \underline{curve}_0) \div B_{3.75}(0) \;=\; E_Q[\, FV(S_{10}\ ;\ \underline{curve}_t) \div \underline{B}_{3.75}(t)\,]\ ;$$

that is, the expectation over the martingale measure is equal to today's value. If we choose $t=3.75$ then $B_{3.75}(3.75) = 1$. Thus, equation (1) implies that:

(2) $$\begin{aligned} FV(S_{10}\ ;\ \underline{curve}_0) &= PV(\text{forward swap today}) \\ &= DF_{3.75} \cdot E_Q[\, FV(S_{10}\ ;\ \underline{curve}_{3.75})\,] \\ &= 0 \end{aligned}$$

where $DF_{3.75} = B_{3.75}(0)$ is the discount factor to year 3.75 and E_Q denotes taking the expectation over the appropriate martingale measure.

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[4] It should also be noted that although this approach solves the problem by a slightly different route than using a full term-structure model, the theoretical foundation is the same. Thus, given similar distributional assumptions, one should arrive at the same answer using either approach.

[5] For a standard CMS/CMT swap the payment is actually made in arrears. This adds a complication to the calculation of the convexity adjustment, which is discussed in detail below.

USING THE PAR SWAP RATE INSTEAD OF THE CURVE

Instead of writing the forward PV as a function of the curve we can write it as a function of a single variable, the (random) par swap rate at year 3.75, $\underline{S}_{10}$:

$$FV(S_{10}\ ;\ \underline{S}_{10})\ .$$

That is, write the PV as:

$$(3)\qquad FV(S_{10}\ ;\ \underline{S}_{10}) = [S_{10} \cdot DC_1\ ] \div [(1+\underline{S}_{10} \cdot DC_1)] \\ + [S_{10} \cdot DC_2\ ] \div [(1+\underline{S}_{10} \cdot DC_1)(1+\underline{S}_{10} \cdot DC_2)] + \ldots \\ + [100 + S_{10} \cdot DC_n\ ] \div [(1+\underline{S}_{10} \cdot DC_1)\cdots(1+\underline{S}_{10} \cdot DC_n)] - 100$$

where

DC_i = day count fraction for period i.

This is simply discounting the fixed cash flows (including the notional principal) by the appropriately defined yield-to-maturity. When this yield-to-maturity is equal to the forward par swap rate, this gives the same value as off the forward curve (i.e. zero). The random variation in the forward curve as of year 3.75 is represented by the variation in the (random) par swap rate $\underline{S}_{10}$, and translated to variations in the price through equation (3).[6]

Now the expectation over the equivalent martingale measure can be written as:

$$(4)\qquad PV(\text{fwd swap}) = DF_{3.75} \cdot E_Q\ [\ FV(S_{10}\ ;\ \underline{S}_{10}\ )\ ]\ .$$

LOG-NORMAL SWAP RATE

The discussion so far applies equally to any assumed distribution for the future par swap rate. For practical application one has to make a specific assumption about the distribution of rates. Often practitioners assume that yields are log-normally distributed. Among others, this assumption has the advantages that

1. Rates remain positive
2. It has more empirical support than the alternative of assuming normal rates (see Coleman, Fisher, Ibbotson (1989)).

[6] Assuming that the future random swap rate is log-normally distributed would not be the same as assuming, for example in a Black-Derman-Toy model, that future short rates are log-normally distributed. It is, however, very close.

When the random par swap rate, $\underline{S}_{10}$, is log-normally distributed then finding the equivalent martingale measure means finding the mean, S_m , such that

(5) $$\int_0^\infty [FV(S_{10};\underline{S}_{10})g(\underline{S}_{10};S_m,s,T)d\underline{S}_{10} = FV(S_{10};curve) = 0$$

where

$g(\underline{S}_{10}; S_m, \sigma, T)$ = log-normal density function with mean S_m, standard deviation σ, and time to reset T

$= \exp[-(ln\underline{S}_{10} - lnS_{10} + \sigma^2T/2)^2 / 2\sigma^2T] / \sqrt{(2\pi\sigma^2T)}$

S_m = mean of the distribution (must be solved for to satisfy 5)

σ = standard deviation (per unit time) of the random swap rate

T = time to the reset .

VALUATION OF THE LINEAR CMS PAYMENT

Once we have the equivalent martingale measure, we can use this measure to value the linear CMS payment. This gives

(6) PV(linear CMS up front) $= DF_{3.75} \cdot E_Q [DC \cdot \underline{S}_{10}] = DF_{3.75} \cdot DC \cdot S_m$

where DC is the day count applied to the payment. We could also define a “convexity spread”: $s = S_m - S_{10}$ and apply it to the forward rate. This would give:

(6’) PV(linear CMS up front) $= DF_{3.75} \cdot DC \cdot (S_{10} + s)$.

CALCULATING THE ADJUSTED MEAN - QUADRATIC APPROXIMATION

To find S_m one must solve equation (5) for S_m . The integral must be evaluated numerically. The is, however, a piece-wise quadratic approximation which allows quick evaluation for the log-normally distributed case. Start by writing the forward value (for receiving fixed) as a portfolio of long a fixed rate bond and short a floating rate bond:

(7) $FV(S_{10} ; y)$ = PV(bond coup = S_{10} , discount rate = y) - 100 .

Then apply a quadratic Taylor series approximation to the price as a function of yield. Since the convexity of a bond is small, this approximation is good over a large range of yields. The approximation is

(8) $$P(y) = P_x + P'_x \cdot [y_x - y] + P''_x \cdot [y_x - y]^2 / 2$$

where

P_x = price of bond at the expansion point
P'_x = first derivative (risk) at expansion point
P''_x = second derivative (convexity) at expansion point

The reason for using this piece-wise quadratic is that it is quick to evaluate the integral over the quadratic. The integral of one piece of the quadratic (from zero to y_x) is

$$(9) \qquad P_x \cdot \int_0^{y_x} g(y)dy \; + \; P'_x \cdot \int_0^{y_x} (y_x - y)g(y)dy \; + \; P''_x \int_0^{y_x} (y_x^2/2 - y \cdot y_x + y^2/2)g(y)dy \; .$$

This is not too difficult, and the result comes to

$$(10)\; P_x \cdot \Phi(\text{-}d_2) \; + \; P'_x \cdot [y_x \Phi(\text{-}d_2) \; - \; y_m \Phi(\text{-}d_1)] \; + \; P''_x \cdot [y_x^2 \Phi(\text{-}d_2) + y_m^2 e^{s^2 T} \Phi(\text{-}d_3)] \; / 2$$
$$- \, P''_x \cdot y_x y_m \Phi(\text{-}d_1)$$

where
T = time to reset (in years)
y_x = yield at expansion point
y_m = adjusted mean of yield distribution
σ = annualized volatility

$$d_1 \; = \; \frac{\ell n(y_m / y_x) + s^2 T/2}{s\sqrt{T}}$$

$$d_2 \; = \; d_1 \; - \sigma\sqrt{T} \qquad\qquad d_3 \; = \; d_1 \; + \sigma\sqrt{T} \; .$$

In general this approximation will not be adequate over the whole range of $y=(0,\infty)$. Multiple expansion points must be used, each resulting in a set of terms such as in (10). These terms can be pieced together and evaluated much faster than the original numerical integration. In particular, the root-finding necessary to solve for S_m is much faster than when evaluating the integral by standard numerical integration.

CMS PAYMENT - in Arrears

In an actual CMS/CMT swap, the payment is made in arrears, by one quarter or one half-year. This complicates the problem relative to payment being made at the reset date (up front). If payment is made up front, the value today is simply the expected value of the linear payment, i.e. the adjusted mean as given in equation (6) above. When the payment is made in arrears, the value must be discounted from payment date back to reset date at the (random) LIBOR rate, then from reset date to today using the discount bond:

(11) $\quad$ PV(linear CMS in arrears) $= DF_{3.75} \cdot DC \cdot E_Q\, [\, \underline{S}_{10} / (1+DC_L \cdot \underline{L}_{3.75})]$

where

$\underline{S}_{10}$ = (random) 10 year swap rate, reset at year 3.75
$\underline{L}_{3.75}$ = (random) LIBOR rate reset at year 3.75, for 3.75 to 4 years
DC = day count fraction for CMS payment
DC_L = day count fraction for LIBOR from reset to payment date.

In other words, the expectation must be taken over the **joint** distribution of the swap rate and the LIBOR rate. This can be done as follows:

1. Calculate the adjusted mean for the swap rate, S_m, as discussed above.
2. Calculate the adjusted mean for the LIBOR rate, in the same manner.
3. Approximate the zero bond $P(\underline{L}_{3.75})=1/(1+DC_L \cdot \underline{L}_{3.75})$ by a piece-wise quadratic in the manner discussed above.
4. Calculate the value of the integral $E_Q\, [\, \underline{S}_{10} \cdot P(\underline{L}_{3.75})]$. Using the piece-wise quadratic approximation to $P(\underline{L}_{3.75})$ will give terms such as $\underline{S}_{10} \cdot \underline{L}_{3.75}$. Because $\underline{S}_{10}$ and $\underline{L}_{3.75}$ are lognormal, their product will also be lognormal.[7]

Because the convexity of a quarterly or semi-annual LIBOR payment will be very small, it is generally possible to use a single quadratic term to approximate $P(\underline{L}_{3.75})$.[8] Taking a Taylor series expansion around the forward rate gives:

$$P(\underline{L}) \approx P_f + P'_f(L_f - \underline{L}) + P''_f(L_f^2 - 2L_f\underline{L} + \underline{L}^2)/2$$

where

P_f = zero evaluated at the forward rate
$\quad = 1/(1+DC_L \cdot L_f)$
P'_f = first derivative evaluated at forward rate
P''_f = second derivative (convexity) evaluated at forward rate

The value of the CMS payment at reset is then:

$$\underline{S}_{10} \cdot P(\underline{L}_{3.75}) \approx \underline{S}_{10} \cdot P_f + P'_f \cdot (\, \underline{S}_{10} \cdot L_f - \underline{S}_{10} \cdot \underline{L}) + P''_f \cdot (\, \underline{S}_{10} \cdot L_f^2 - \underline{S}_{10} \cdot 2L_f\underline{L} + \underline{S}_{10} \cdot \underline{L}^2)/2 \ .$$

Taking the expectation over the equivalent martingale measure gives:

$$(12) \quad E_Q\, [\, S_{10} \cdot P(\underline{L}_{3.75})\,] \approx S_m \cdot P_f + S_m \cdot P'_f \cdot (L_f - L_m e^{r s s_\ell T}) + S_m \cdot P''_f \cdot (L_f^2 - 2L_f L_m e^{r s s_\ell T} + L_m^2 e^{(s_\ell^2 + 2 r s s_\ell)T})/2$$

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[7] With the inclusion of the LIBOR rate, both the volatility of the LIBOR rate and the correlation between the swap and LIBOR rate will enter.

[8] One can use the third derivative to estimate the error of the quadratic approximation. For a semi-annual payment with rates at 8%, rates would have to move about 2,000bp for the approximation error in the quadratic to rise to 0.1%. For a reset at 10 years, with annual volatility at 20%, a rise of 2,000bp is a rise of about 2 standard deviations; i.e. far out in the probability distribution.

Using the same quadratic approximation, the adjusted LIBOR mean L_m can be calculated as the solution to the quadratic equation:

$$L_m^2 \cdot P''_f \cdot e^{s^2 T}/2 \;-\; L_m \cdot(P'_f \;+ P''_f \cdot L_f) \;+ P''_f \cdot L_f^2 / 2 \;+ P'_f \cdot L_f \;=\; 0 \;.$$

Equation (12) can also be written as:

(13) $$E_Q [\, \underline{S}_{10} \cdot P(\underline{L}_{3.75}) \,] \;\approx P_f S_m \cdot [\, 1 + P'_f \cdot(L_f - L_m e^{\,rss_\ell T}) / P_f + P''_f \cdot(L_f^2 - 2L_f L_m e^{\,rss_\ell T} + L_m^2 e^{(s_\ell^2 + 2rss_\ell)T})/2P_f \,]$$

where

σ = volatility of forward swap rate
σ_ℓ = volatility of forward LIBOR rate
ρ = correlation between forward rates .

The advantage of writing it in this form is that the PV is expressed as an adjusted rate or certainty equivalent cash flow multiplied by a discount factor from the forward curve. In other words, the final convexity adjusted rate can be discounted back directly using the forward curve. The net convexity adjustment (including the effect of payment in arrears) can also be expressed as a spread to the forward rate:

(14) $$s \;= S_m \cdot [\, 1 + P'_f \cdot(L_f - L_m e^{\,rss_\ell T}) / P_f + P''_f \cdot(L_f^2 - 2L_f L_m e^{\,rss_\ell T} + L_m^2 e^{(s_\ell^2 + 2rss_\ell)T})/2P_f \,] \;-\; S_{10}$$

LIBOR-IN-ARREARS ADJUSTMENT

The idea for LIBOR-in-arrears is very much the same, but without the problem of payment in arrears. Referring back to figure 2 where an annual LIBOR-in-arrears payment of $\underline{L}_5 \cdot AD/360$ is made at year 5, the hedge is a standard FRA which pays $\underline{L}_5 \cdot AD/360 \div (1+\underline{L}_5 \cdot AD/360)$ at year 5. In other words, the present value of the (traded) 5 into 6 year FRA is

(15) $$PV(FRA) \;=\; DF_5 \cdot (y_5 - r)/(1+y_5) \;=\; DF_5 \cdot [\, 1 - (1+r)/(1+y_5) \,] \,,$$

where

DF_5 = Discount factor for fixed cash flows occurring at year 5

y_5 = today's forward rate from year 5 to year 6 (adjusted for day count; i.e.

$y_5 = L_5 \cdot AD/360$)

r = agreed (fixed) FRA rate, adjusted for day count.

Alternatively, the PV can be thought of as the expectation (over the equivalent martingale measure) of the random LIBOR as of year 5:

(16) $$PV(FRA) = DF_5 \cdot E_Q [(\underline{y}_5 - r)/(1+\underline{y}_5)] = DF_5 \cdot E_Q [1 - (1+r)/(1+\underline{y}_5)] ,$$

where

$\underline{y}_5$ = random LIBOR (as of year 5) from year 5 to year 6, adjusted for day count

$E_Q [\cdot]$ = expectation taken over the equivalent martingale measure.

Once one has the adjusted mean one can calculate a spread $s = y_m - y_f$ and apply it to the forward rate; i.e. use the adjusted mean instead of the forward rate as the projected payment:

$$PV(\text{linear LIBOR}) = DF_5 \cdot E_Q [\underline{y}_5] = DF_5 \cdot y_m = DF_5 \cdot (y_f + s) .$$

EXAMPLE OF CMS SWAP

Figure 5 shows the implied forward rates and adjusted mean rates for a 10 year swap against 10 year CMS, assuming annual payments, a flat 7.5%ab forward rate, and 15% volatility.[9] With no convexity adjustment, the break-even fixed side rate would be 7.5%, since the forward curve is flat. At a 15% volatility, however, the convexity adjustment increases the break-even rate to 7.72%; i.e. by 19bp. In other words, a swap would be LIBOR flat versus CMS less 19bp (in this case the effect of the curve and the day count basis are both zero).[10]

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[9] Most CMT/CMS swaps are quarterly payments and resets. Annual payments and resets are used here to simplify the example. Note, however, that the effect of payment in arrears is larger here than would be the case for quarterly payments.

[10] It is worth noting, in passing, that Brotherton-Ratcliffe and Iben put forward an approximation for the convexity spread:

$$s = r^2 \cdot \sigma^2 \cdot T \cdot P'' / (2 \cdot P')$$

where

s = convexity adjustment (spread in percent, i.e. 0.01 is 1bp)
r = forward rate (in percent, as in 8.0)
σ = volatility (in decimal, as in 0.15)
T = time to reset (payment) date in years
P' = first derivative of price w.r.t. yield

Figure 5 - Adjusted Mean Rates for Annual Resets Against 10 Year CMS, 7.5%ab curve, 15% Volatility

Reset date	0	1	2	3	4	5	6	7	8	9
Adjusted rate	7.50	7.55	7.59	7.64	7.69	7.74	7.79	7.84	7.89	7.94
Convx sprd (bp)	0.0	4.6	9.4	14.1	19.0	23.9	28.9	33.9	39.1	44.2
Payment date	1	2	3	4	5	6	7	8	9	10
PV of sprd (bp)	0.0	4.0	7.5	10.6	13.2	15.5	17.4	19.0	20.4	21.5

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P" = second derivative of price w.r.t. yield

They arrive at this approximation by applying a heuristic argument to the convexity cost for CMS swaps. This approximation is reasonably good but does not take into account the payment in arrears. For the example from figure 5, their approximation gives a spread 5.8bp for the year 1 reset versus 4.6bp in the table, and 52.0 for the year 9 reset versus 44.2 in the table. Using the equivalent martingale approach the reset at year nine with payment up-front would be 56.3 and with payment one quarter in arrears would be 53.0.